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ONE-DIMENSIONAL ENERGY TRANSFER IN RADIANT MEDIA†

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Abstract—The equations of one-dimensional radiative energy transfer are extended from their classical astrophysics form to include walls of arbitrary radiative properties. The concepts of emissivity and penetration length are examined. As an application, the case of the steady infinite flat layer is considered, with conduction and radiation present. The wall conditions are so chosen as to give a good model of a low-speed high-temperature boundary-layer. It is found that the effect of the "long-range" process of radiation is to smooth out the temperature profiles and relieve the sharp temperature gradients at the cool wall. As a result, the application of the exact method yields a lower value of both components of the total heat flux (radiation plus convection) than calculated previously by assuming a temperature profile on the basis of conduction only. Such coupling of convective and radiative fluxes is governed by the magnitude of a non-dimensional parameter, depending on the physical properties and the flow geometry of the problem.

Résumé—Les équations de la transmission unidimensionnelle d'énergie par rayonnement ont été étendues, à partir de leur forme classique en astrophysique, de façon à traiter le cas de parois à propriétés de rayonnement quelconques. Les notions d'émissivité et de longueur de pénétration sont examinées. Le cas d'une couche infinie, plane et permanente, en présence de conduction et de rayonnement, est traité à titre d'application. Les conditions de paroi sont choisies de façon à donner un bon modèle de couche limite à basse vitesse et à haute température. On trouve que l'effet à grande distance du processus de rayonnement est d'adoucir les profils de température et de mettre en relief les importants radients de température à la paroi froide. Il en résulte que l'application de la méthode exacte fournit pour les deux composantes du flux de chaleur total (rayonnement + convection) une valeur plus basse que celle calculée précédemment en supposant le profil de température dû à la seule conduction. Une telle combinaison des flux de convection et de rayonnement est déterminée par la grandeur d'un paramètre sans dimensions dépendant des propriétés physiques et de la géométrie de l'écoulement du problème.

Zusammenfassung—Die Gleichungen für die eindimensionale Energieübertragung durch Strahlung wird von ihrer klassischen Form der Astrophysik erweitert, um auch Wände mit beliebigen Strahlungseigenschaften einzuschließen. Die Begriffe der Emission und der Durchdringungslänge werden untersucht. Zur Anwendung wird eine unendliche flache Schicht im Beharrungszustand betrachtet, in der Leitung und Strahlung stattfindet. Die Wandbedingungen werden so gewählt, dass ein brauchbares Modell für eine Hochtemperaturgrenzschicht bei kleinen Geschwindigkeiten entsteht. Es zeigt sich, dass die Wirkung der Strahlung die Temperaturprofile ausgleicht und den steilen Temperaturgradienten an der kalten Wand abflacht. Die Anwendung der exakten Methode ergibt geringere Werte für beide Komponenten des gesamten Wärmestroms (Strahlung und Konvection), als sie sich bei der Annahme eines Temperaturprofiles für reine Leitung bisher ergeben hat. Das Zusammenwirken der Wärmeströme durch Konvektion und Strahlung wird durch eine dimensionslose Kenngrösse bestimmt, die von den Stoffwerten und der geometrischen Anordnung abhängt.

Abstract—Классические уравнения одномерного лучистого переноса энергии, используемые в астрофизике распространяются на случай лучистого теплообмена в ограниченном пространстве с произвольными радиационными характеристиками стенок. Рассматриваются понятия степени черноты тела и глубины проникновения. В качестве примера рассматривается случай стационарного неограниченного плоского слоя при наличии теплопроводности и лучистого теплообмена. Условия на стенках выбраны таким образом, чтобы получить хорошую модель низкоскоростного высокотемпературного пограничного слоя. Установлено, что влияние „дальнодействующего"

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процесса излучения заключается в сглаживании температурных профилей и уменьшении „остроты" температурных градиентов на холодной стенке. Применение точного метода даёт более низкую величину обеих составляющих полного потока тепла (лучистой и конвективной) нежели это вычислялось прежде, когда предполагалось, что температурный профиль определяется только теплопровоностью. Для данной задачи сочетание конвективного и лулцстого потоков регулируется величиною безразмерного параметра, зависящего от физических свойств и геометрии потока.

NOTATION

$A(B)$,	equation (34);
B, B_ν,	Planck's function (equations (6) and (7);
C,	variable coefficient (equation (11));
c,	velocity of light;
E, E_ν,	radiative energy;
$E_n(t)$,	equation (19);
h,	Planck's constant;
k,	conduction heat transfer coefficient;
I, I_ν,	specific intensity (equation (1));
j_ν,	emission coefficient;
L,	plane layer thickness;
$\mathbf{L}$,	direction of incident radiation;
l,	penetration length;
N_{r-c},	equation (37);
$\mathbf{n}$,	unit vector normal to surface $d\sigma$ (Fig. 1);
q,	radiative energy flux;
r,	reflectivity;
ds,	elementary length along $\mathbf{L}$;
t,	integration variable (also: time, in equation (1));
T,	absolute temperature;
tr,	transparency;
V,	flight velocity;
y,	vertical co-ordinate (Fig. 2);
α,	absorptivity;
ε,	emissivity;
θ,	spherical co-ordinate of $\mathbf{L}$ (Fig. 1);
κ,	absorption coefficient;
μ,	equation (9);
ν,	frequency;
σ,	Stefan's constant;
$d\sigma$,	surface element (Fig. 1);
ρ,	density;
τ, τ_ν,	optical thickness (equation (8));
ϕ,	spherical co-ordinate of $\mathbf{L}$ (Fig. 1);
$d\omega$,	solid angle (Fig. 1).

Subscripts

g,	gas;
w,	wall;
ν,	frequency;
0,	lower wall;
2, L,	upper wall.

Superscript

*,	floating boundary.

INTRODUCTION

ENERGY transfer by radiation has been for a long time a familiar problem to the physicists concerned with high-temperature gases. Its important role in specialized subjects of applied physics, such as blast waves, plasma physics and astrophysics has been the object of many studies and these different domains have attained a high degree of organization [1, 2, 11, 15].

Technological progress has, meanwhile, steadily increased the demand for engineering designs capable of withstanding higher and higher temperatures. The associated problems of radiative energy transfer are becoming, therefore, increasingly important in the field of combustion [4, 5] and propulsion. They are also to be considered, now, in very high speed aerodynamics [16] and new reactor concepts.

A major difference between the two classes of problems just described is due to the existence of solid boundaries in most engineering systems using radiative media. In this paper, the mathematical expressions of radiant energy transfer developed in astrophysics will be extended to include wall effects in simple one-dimensional geometries. A typical application will be made to a steady infinite flat layer, with wall conditions so chosen as to give a good model of a low-speed high-temperature boundary layer.

A. ONE-DIMENSIONAL RADIATION FLUX

Specific intensity I_ν and flux q_ν

The fundamentals of the theory of radiation transfer in gases can be found in standard textbooks of astrophysics [1, 2]. Two important quantities to be used extensively in this paper will be redefined here. The notation adopted is that of Kourganoff [3].

Let dE_ν be the amount of radiative energy in the frequency interval $\nu, \nu + d\nu$ transmitted through an elementary surface $d\sigma$ at a point P during a time interval dt within a solid angle $d\omega$ (see Fig. 1). The normal **n** to the surface and

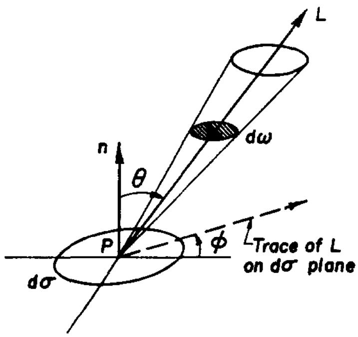

FIG. 1. Radiation intensity symbols.

the direction **L** of the solid angle form an angle θ. I_ν, the *specific intensity* in the direction **L** is defined as:

$$I_\nu = \lim. \left| \frac{dE_\nu}{\cos\theta \, d\sigma \, d\omega \, d\nu \, dt} \right|_{d\sigma,\, d\omega,\, d\nu,\, dt \to 0} \quad (1)$$

Accordingly, I_ν depends on the location of P and on the direction **L**. Because of the introduction of $\cos\theta$ in equation (1), the specific intensity I_ν is usually independent of the angle θ between the direction **L** and the normal **n** to the elementary surface. It should be pointed out that this definition is *not* the one most frequently used in engineering ([17], equation 13–6), where $\cos\theta$ does not appear in the definition of the intensity. As a result, "engineering" radiation intensities vary with θ (e.g. Lambert's law). In this paper, the "astrophysics" definition of the intensity will be used (equation (1)).

The flux q_ν of radiative energy across the surface $d\sigma$, in the frequency interval $d\nu$, is obtained simply by summing up the quantity

$$\frac{dE_\nu}{d\sigma \, dt \, d\omega} = I_\nu \cos\theta \, d\omega$$

for all the directions **L** ($d\omega = \sin\theta \, d\theta \, d\phi$). It is convenient to split the net flux q_ν into the contribution q_ν^+ coming from the side of the normal unit vector **n** and the contribution q_ν^- from the opposite side:

$$q_\nu^+ = \int_0^{\pi/2} \int_0^{2\pi} I_\nu(\theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi \quad (2)$$

$$q_\nu^- = -\int_{\pi/2}^{\pi} \int_0^{2\pi} I_\nu(\theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi \quad (3)$$

Equation of radiative energy transfer

The intensity, on a length ds, along the direction **L**, is attenuated by absorption and scattering away from the direction **L**, while it is reinforced by the energy emitted by the particles along **L** and the scattering of photons from other directions into the direction **L**.

The intensity I_ν is therefore determined by the equation:

$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho I_\nu + \rho j_\nu \quad (4)$$

where κ_ν and j_ν are, respectively, the absorption and emission coefficients [3]. It is further shown in [2, 3], that, provided the gradients of temperature are not too considerable and the densities not too small, "local thermodynamic equilibrium" can be assumed and equation (4) can be written:

$$-\frac{1}{\rho\kappa_\nu} \frac{dI_\nu}{ds} = I_\nu - B_\nu \quad (5)$$

where B_ν is Planck's function [2]:

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp(h\nu/kT) - 1} \quad (6)$$

with

$$B = \int_0^\infty B_\nu \, d\nu = \frac{\sigma}{\pi} T^4 \quad (7)$$

Specific intensity in one-dimensional problems

A glance at equations (2) and (5) shows that all radiation problems but those with the simplest geometry, will be difficult to solve. In this study, only one-dimensional geometries will be discussed.

It is convenient to define at this point the *optical thickness* τ_ν such that:

$$d\tau_\nu \equiv \rho\kappa_\nu \, dy \quad (8)$$

where y is the privileged direction of the one-dimensional problem illustrated on Fig. 2. If θ is the angle between the direction $\mathbf{L}$ and the negative (downward) y-direction, then:

$$dy = -\mu \, ds \text{ where } \mu \equiv \cos\theta \tag{9}$$

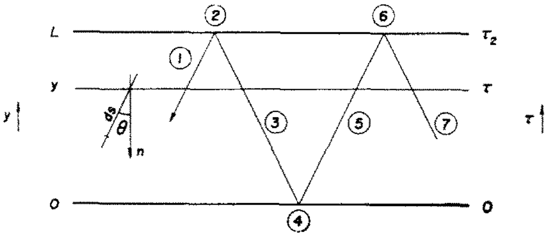

FIG. 2. Multiple reflexions in a plane layer.

Equation (5) can then be rewritten:

$$\mu \frac{dI}{d\tau_\nu} = I_\nu - B_\nu \tag{10}$$

The solution $I_\nu(\tau_\nu)$ of equation (10) is simply obtained by the variable coefficient method:

$$I_\nu = C(\tau_\nu) \exp(\tau_\nu/\mu) \tag{11}$$

The function $C(\tau_\nu)$ is, after substitution of equation (11) into equation (10):

$$C(\tau_\nu) = C(\tau_\nu^*) - \int_{\tau_\nu^*}^{\tau} \frac{B_\nu(t)}{\mu} \exp(-t/\mu)\, dt \tag{12}$$

where τ_ν^* is an arbitrary value, to be determined by the boundary value of the problem. For instance, the radiation directed upwards in Fig. 2 will be determined by the condition at the lower wall: $\tau_\nu^* = 0$. After substitution, equation (11) becomes:

$$I_\nu^- = \int_0^{\tau_\nu} B_\nu(t) \exp\{-(t-\tau_\nu)/\mu\} \frac{dt}{\mu} + \\ + I_\nu^-(0) \exp(\tau_\nu/\mu) \tag{13}$$

Similarly, the radiation directed downwards will be determined by the conditions at the upper wall:

$$I_\nu^+ = \int_{\tau_\nu}^{\tau_{\nu 2}} B_\nu(t) \exp\{-(t-\tau_\nu)/\mu\} \frac{dt}{\mu} + \\ + I_\nu^+(\tau_{\nu 2}) \exp\{(\tau_\nu - \tau_{\nu 2})/\mu\} \tag{14}$$

Radiation flux in one-dimensional problems

Since we confine ourselves to one-dimensional problems, equations (2) and (3) can be integrated with respect to ϕ. One obtains:

$$q_\nu^+ = 2\pi \int_0^1 I_\nu(\mu)\, \mu d\mu \tag{15}$$

$$q_\nu^- = 2\pi \int_0^{-1} I_\nu(\mu)\, \mu d\mu \tag{16}$$

It is then a simple matter to substitute equations (13) and (14) into (15) and (16). The result is:

$$q_\nu^- = 2\pi \int_0^{\tau_\nu} B_\nu(t)\, E_2(\tau_\nu - t)\, dt + \\ + 2q_\nu^-(0)\, E_3(\tau_\nu) \tag{17}$$

$$q_\nu^+ = 2\pi \int_{\tau_\nu}^{\tau_{\nu 2}} B_\nu(t)\, E_2(t - \tau_\nu)\, dt + \\ + 2q_\nu^+(\tau_{\nu 2})\, E_3(\tau_{\nu 2} - \tau_\nu) \tag{18}$$

where the dependence on μ appears through the functions:

$$E_n(t) \equiv \int_0^1 \mu^{n-2} e^{-t/\mu}\, d\mu \tag{19}$$

These functions are conveniently tabulated by Kourganoff [3]. The net flux q_ν is the difference between equations (17) and (18).

To obtain the total flux, it is necessary to integrate the flux expressions for all wavelengths. To simplify the discussion, the medium under study will be assumed to be *gray*: by definition, the absorption coefficient κ_ν will then be independent of the frequency ν and so will be the optical thickness τ_ν. The only wavelength-dependent function left in the expressions derived above, will be Planck's function (equation (6)). It can therefore be integrated separately (equation (7)), and the subscript ν will be dropped from the rest of the paper.

B. WALL EFFECTS†

The second terms, on the right-hand side of the expressions (17) and (18) of q^- and q^+,

† Part B of this paper is a condensation of a more detailed study made by the Senior Author in Report No. A–59–8, School of Aeronautical Engineering, Purdue University, Lafayette, Indiana.

account for the contribution of the wall to the radiative flux. Whenever walls are non-existent, as it is the case in star photospheres, this term disappears and the expressions (17) and (18) take their classical Milne formulation [3].

In general, however, the contribution from the wall is determined by its absorptivity a_w, its emissivity ϵ_w, its transparency tr_w and its reflectivity r_w. The relations between these quantities are:

$$a_w + tr_w + r_w = 1 \tag{20}$$

$$a_w = \epsilon_w \tag{21}$$

The flux *downwards* q^+ in the slab geometry shown on Fig. 2, is at a given station τ, made of several components:

(1) the flux from the gas slab $\tau - \tau_2$

$$2\pi \int_\tau^{\tau_2} B(t)\, E_2(t-\tau)\, dt$$

(2) the flux from the upper wall, attenuated

$$2\pi \epsilon_2 B(\tau_2)\, E_3(\tau_2 - \tau)$$

(3) the flux from the gas slab $0 - \tau_2$, after reflection on the upper wall and attenuation

$$\left\{ 2\pi \int_0^{\tau_2} B(t)\, E_2(t-\tau_2)\, dt \right\} 2r_2\, E_3(\tau_2 - \tau)$$

(4) the flux from the lower wall, after two attenuations and one reflection:

$$2\pi\, \epsilon_0 B(0)\, E_3(\tau_2 - 0)\, r_2\, 2E_3(\tau_2 - \tau)$$

(5) the flux from the gas slab $0 - \tau_2$, after two reflections and two attenuations:

$$\left\{ 2\pi \int_0^{\tau_2} B(t)\, E_2(t)\, dt \right\} \times$$
$$\times\ r_0\, 2E_3(\tau_2)\, r_2\, 2E_3(\tau_2 - \tau)$$

(6) the flux from the upper wall twice reflected and three times attenuated

$$2\pi\, \epsilon_2 B(\tau_2)\, E_3(\tau_2) r_0\, 2E_3(\tau_2) r_2\, 2E_3(\tau_2 - \tau)$$

(7) . . . ., etc., . . .

It is to be noted that the contribution from the upper wall in (6) is equal to its contribution in (2), except for the attentuation factor:

$$4r_0 r_2\, E_3^2(\tau_2)$$

due to two additional reflections. The contribution of the wall along a ray which has been reflected $2n$ times, is similarly attenuated by a factor:

$$[4r_0 r_2\, E_3^2(\tau_2)]^n$$

The total contribution of the upper wall is therefore equal to:

$$2\pi\, \epsilon_2 B(\tau_2)\, E_3(\tau_2 - \tau) \left[\sum_{n=0}^{n=\infty} \{4r_0 r_2\, E_3^2(\tau_2)\}^n \right]$$

which can be rearranged $\{4r_0 r_2\, E_3^2(\tau_2) < 1\}$ as:

$$\frac{2\pi}{1 - 4r_0 r_2\, E_3^2(\tau_2)}\, \epsilon_2 B(\tau_2)\, E_3(\tau_2 - \tau) \tag{22}$$

Using the same method to account for the other contributions to the flux, the expression of the *net* flux across station τ is therefore:

$$\begin{aligned} q = q^+ - q^- = {} & 2\pi \int_\tau^{\tau_2} B(t)\, E_2(t-\tau)\, dt - \\ & - 2\pi \int_0^\tau B(t)\, E_2(\tau - t)\, dt + \\ & + \frac{2\pi}{1 - 4r_0 r_2\, E_3^2(\tau_2)} \Big\{ \epsilon_2 B(\tau_2) E_3(\tau_2 - \tau) - \\ & - \epsilon_0 B(0)\, E_3(\tau) + \\ & + \epsilon_0 B(0)\, E_3(\tau_2)\, 2r_2\, E_3(\tau_2 - \tau) - \\ & - \epsilon_2 B(\tau_2)\, E_3(\tau_2)\, 2r_0\, E_3(\tau) + \\ & + 2r_2\, E_3(\tau_2 - \tau) \int_0^{\tau_2} B(t)\, E_2(t - \tau_2)\, dt - \\ & - 2r_0\, E_3(\tau) \int_0^{\tau_2} B(t)\, E_2(t)\, dt + \\ & + 4r_0 r_2\, E_3(\tau_2)\, E_3(\tau_2 - \tau) \int_0^\tau B(t)\, E_2(t)\, dt - \\ & - 4r_0 r_2\, E_3(\tau_2)\, E_3(\tau) \int_0^{\tau_2} B(t)\, E_2(t - \tau_2)\, dt \Big\} \end{aligned} \tag{23}$$

This lengthy expression is the most general form of the flux through a one-dimensional non-scattering medium†. Among its many

† A completely general form, including non isotropic scattering, is under preparation by R. Viskanta, School of Mechanical Engineering, Purdue University.

applications to special cases are familiar formulas.

(a) Slab of very large thickness (star photosphere or blast waves):

$$\epsilon_0 = \epsilon_2 = r_0 = r_2 = 0 \quad \tau_2 \to \infty$$

$$q = 2\pi \int_{\tau}^{\infty} B(t)\, E_2(t-\tau)\, dt - - 2\pi \int_{-\infty}^{\tau} B(t)\; E(t)\; dt$$

equation (11–4) of [3].

(b) Transparent medium between two gray parallel plates:

$$\kappa \sim 0 \to \tau \sim 0,\, E_3(\tau) \sim E_3(0) = \tfrac{1}{2},\, \epsilon = 1 - r$$

$$q = \sigma\,(T_2^4 - T_0^4)\, \frac{1}{1/\epsilon_2 + 1/\epsilon_0 - 1}$$

equation (4–5) of [5].

(c) Absorbing medium between two black body plates at same temperature T_w:

$$r_0 = r_2 = 0\,,\ \epsilon_0 = \epsilon_2 = 1$$

$$q = 2\pi \int_{\tau}^{\tau_2} B(t)\, E_2(t-\tau)\, dt - - 2\pi \int_{0}^{\tau} B(t)\, E_2(t)\, dt + + 4\pi\, B(\tau_2)\, E_3(\tau_2 - \tau) - 4\pi\, B(0)\, E_3(\tau)$$

Assume also $\tau \ll 1$; then $E_2 \simeq 1$, $E_3 \simeq 1/2 - t$:

$$q = 2\int_{y}^{L} \sigma T^4\, \rho\kappa dy - 2\int_{0}^{y} \sigma T^4 \rho\kappa dy + + \sigma(T_2^4 - T_0^4) - 2\sigma T_2^4 \rho\kappa\,(L-y) + 2\sigma T_0^4 \rho\kappa y$$

At the lower wall, $y = 0$, $\tau = 0$, and since $T_0 = T_2 = T_w$,

$$q_w = 2\int_{0}^{L} \sigma T^4 \rho\kappa dy - 2\sigma T_w^4\, \rho\kappa L$$

and, since $2\rho\kappa L = \epsilon$ (equation (27)), this is equivalent to:

$$q_w = \sigma\,(\epsilon_g T_g^4 - a_g T_w^4)$$

equation (4–57) of [5].

C. PHYSICAL PROPERTIES

With these expressions of fluxes and intensities available, it is now possible to describe the physical properties of the medium, in a form more directly applicable to transfer problems.

Emissivity ϵ of a constant-temperature gas layer†

A simple application of equation (23) consists of the constant-temperature slab, with a transparent upper wall and either a cool [$B(0) \simeq 0$], black-body ($r_0 = 0$) lower wall, or a transparent lower wall. The expression of the radiant flux on either face of the slab, is therefore:

$$q = 2\pi \int_{0}^{\tau_2} B(t)\, E_2(t)\, dt \tag{24}$$

and

$$q = 2\sigma T^4 \int_{0}^{\tau_2} E_2(t)\, dt = -\, 2\sigma T^4 \int_{0}^{\tau_2} d\,[E_3(t)]$$

and since $E_3(0) = \frac{1}{2}$, [see (31a)]

$$q = \sigma T^4\,[1 - 2\,E_3(\tau_2)] \tag{25}$$

If, furthermore, the medium is optically thin (i.e. $\tau \ll 1$), $E_3(\tau)$ can be written in good approximation (31a):

$$E_3(\tau) \simeq \tfrac{1}{2} - \tau$$

Hence:

$$\begin{aligned} q &= \sigma T^4\{1 - 2(\tfrac{1}{2} - \tau_2)\} \\ &= 2\tau_2 \sigma T^4 \\ &= 2\rho\kappa L \sigma T^4 \quad (\tau_2 \ll 1) \end{aligned} \tag{26}$$

where L is the physical thickness of the layer corresponding to τ_2. It is then natural to call the quantity $2\rho\kappa L$ the emissivity ϵ of the gas layer:

$$\frac{\epsilon}{L} \equiv 2\rho\kappa \tag{27}$$

This convenient ratio has been tabulated, for instance, in [6] for high-temperature air. The coefficient κ can be obtained directly from this relation.

† This derivation of the relationship between ϵ and κ was suggested by W. Glauz, School of Engineering Sciences, Purdue University. It avoids the difficulty met in [6], in which the contribution of grazing rays in the slab (very large $\rho\kappa s$), is used in an "optically thin" approximation (small $\rho\kappa s$).

Penetration length l

In the case studied above, equation (25) shows that, if there is a transparent or no upper wall a black-body radiation flux to the lower wall will be achieved only for $\tau_2 \rightarrow \infty$ (where $E_3 \rightarrow 0$). The intensity variation along a beam of arbitrary angle (arbitrary fixed μ) is, at the lower wall ($\tau = 0$) according to equation (14):

$$I^+ = \frac{\sigma T^4}{\pi}\int_0^{\tau_2} \exp\left(-t/\mu\right)\frac{dt}{\mu} =$$

$$\frac{\sigma T^4}{\pi}\left\{1 - \exp\left(-\tau_2/\mu\right)\right\} \quad (28)$$

and, for optically thin layers (τ_2 1),

$$I^+ = \frac{\sigma T^4}{\mu\pi}\tau_2 = \rho\kappa L\frac{\sigma T^4}{\pi\mu} \quad (29)$$

It can be concluded from these two equations that both I and q depend on the absorption properties of the medium, only through the optical thickness τ. The optical thickness is therefore a useful dimensionless concept and is refered to often as the Bueger number (N_{Bu}) in the Russian literature [7].

A useful index of the absorption properties of a substance is also the physical thickness l corresponding to an optical thickness unity:

$$l \equiv \frac{1}{\rho\kappa}$$

This "penetration length" l has been discussed in [8]. It is shown in Fig. 3 for high-temperature air in terms of temperature and density ratios, using the values of ϵ from [6].

If the characteristic dimension L of the problem under study is much less than l, then

$$\tau = \frac{L}{l} \ll 1$$

and the "optically thin" layer approximation of equation (29) can be used. We note that in equation (29), the contributions of all the infinitesimal layers forming a layer of finite thickness L, are *additive*. Physically, this means that in optically thin layers, the energy radiated in any part of the layer is *not* absorbed by the other parts. This is a great simplification in some transfer problems in re-entry aerodynamics, where for most altitudes ($\rho \ll \rho_0$), velocities ($V < 35{,}000$ ft/sec), and vehicle sizes ($L < 100$ cm): $L \ll l$ as illustrated in Fig. 3.

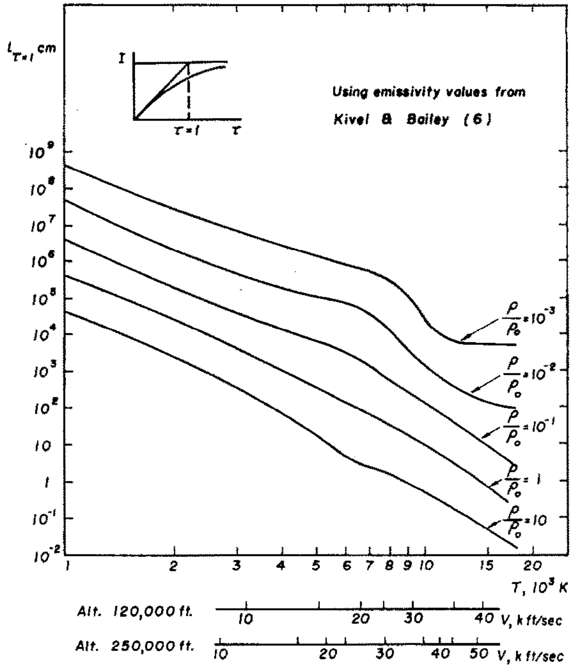

FIG. 3. Length of penetration $l = 1/\rho\kappa$ in aerodynamic problems.

On the contrary, if $l \ll L$, intermediate absorption takes place and at the limit, l plays the role of a mean free path (Rosseland), leading to a differential form for the flux expression q. This is often a considerable mathematical simplification which is justified in many cases [5, 7, 10], including high temperature ($T > 10{,}000$°K), large size ($L \sim 10^4$ cm) blast waves in air [11].

D. THE PLANE-LAYER PROBLEM

The determination of the temperature profile and energy flux across an infinite plane layer of uniform thickness and arbitrary wall temperature, is a classical problem of conduction heat transfer [4], even when chemical reactions occur in the layer or at the walls [14]. The energy conservation equation to solve is then in purely differential form.

The energy conservation equation

This problem is greatly complicated, however, when radiation contributes appreciably to the energy transfer. In this case a complex integral term, equation (23), is added to the convection flux. Unless this new term can be reduced to a differential form, as for some cases discussed in Section C, no closed solution is available to this class of problems.

In this paper, a numerical solution to the steady-state plane-layer problem is presented. The solution can be applied to the fluid-layer problem (Couette flow) if the velocity gradients are not too large (no energy dissipation term in the energy equation). Such a simple model, showing a close analogy with the boundary-layer problem in high-temperature flow, is considered here. The upper wall is transparent (emissivity $\epsilon_2 = 0$, reflectivity $r_2 = 0$), and the lower wall is an opaque gray surface (emissivity ϵ_0, reflectivity $r_0 = 1 - \epsilon_0$). Equation (23) reduces, in this case, to:

$$q = 2\pi \int_{\tau}^{\tau_2} B(t)\, E_2(t-\tau)\, dt - \\ - 2\pi \int_0^{\tau} B(t)\, E_2(\tau - t)\, dt - 2\pi\epsilon_0\, B(0)\, E_3(\tau) - \\ - 4\pi r_0\, E_3(\tau) \int_0^{\tau_2} B(t)\, E_2(t)\, dt \qquad (30)$$

The physical meaning of the four terms on the right-hand side of equation (30) is, respectively:

(a) the energy radiated past the elementary slab τ by all the elementary slabs located above ($\tau < t < \tau_2$);
(b) the energy radiated past the elementary slab τ by all the elementary slabs located below ($0 < t < \tau$);
(c) the fraction of energy radiated by the lower wall that reaches the layer τ, the other fraction being absorbed by the layer $(0 - \tau)$;
(d) the fraction of the energy radiated by the slab to the lower wall, after partial reflection by the lower wall and partial absorption of the layer $(0 - \tau)$.

Substitution of this value of q into the energy conservation equation for the one-dimensional steady state:

$$k \frac{\partial T}{\partial y} + q = \text{constant} \qquad (31)$$

yields a non-linear integro-differential equation that is satisfied by a temperature distribution $T(y)$ to be determined.† In the general non-gray case, this same method would apply with an additional integration of the radiative terms for all wavelengths.

The solution of the aerodynamic flow problem

A further (but not essential to the solution) simplification to the aerodynamic problems in radiant media is due to the low optical thicknesses τ involved (i.e. high penetration lengths l, as seen on Fig. 3). In this case, Kourganoff ([3], p. 255) shows that

$$\begin{aligned} E_2(t) &= 1 - 0(t) \\ E_3(t) &= \tfrac{1}{2} - t + 0(t^2) \end{aligned} \qquad (31a)$$

where $0(t^n)$ means "terms of order n and higher". Substituting into equations (30) and (31), one sees that the contribution of the variable part of $E_2(t)$ is a second-order term in t, which can be neglected.

The physical meaning of this simplification has already been found in the preceding section: in an optically thin layer, the energy radiated by any elementary thickness of the layer is *not* absorbed by the rest of the layer. q can then be written:

$$q = 2\pi \int_{\tau}^{\tau_2} B(t)\, dt - 2\pi \int_0^{\tau} B(t)\, dt - \\ - 2\pi\epsilon_0\, B(0)\, (\tfrac{1}{2} - \tau) - 2\pi r_0 \int_0^{\tau_2} B(t)\, dt$$

and since $r_0 = 1 - \epsilon_0$ (opaque wall),

$$q = 2\pi\epsilon_0 \int_0^{\tau_2} B(t)\, dt - \pi\epsilon_0\, B(0) - \\ - 4\pi \int_0^{\tau} B(t)\, dt + 2\pi\epsilon_0\, B(0)\tau$$

† Once the distribution $T(\tau)$ is established, $T(y)$ is easily obtained through the relation $d\tau = \rho\kappa\, dy$.

or:

$$q = 2\pi\epsilon_0 B(0)\tau - 4\pi \int_0^\tau B(t)\, dt + \text{constant}$$

After substitution into equation (31), the energy transfer equation is:

$$k \frac{\partial T}{\partial y} + 2\pi\epsilon_0 B(0)\,\tau - 4\pi \int_0^\tau B(t)\, dt = C \qquad (32)$$

where C is a constant. It is left now to determine the profile $T(y)$ which satisfies this question. It can be further simplified if the optical parameters are used:

$$dB = \frac{1}{\pi} 4\sigma T^3\, dT \qquad (33)$$

$$d\tau = \rho\kappa dy$$

After substitution in (33):

$$A(B) \frac{dB}{d\tau} - 4\pi \int_0^\tau B(t)dt + 2\pi\epsilon_0 B(0)\tau = C \qquad (34)$$

where $A = k\pi(\rho\kappa/4\sigma T^3)$, hence a function of B through T.

This equation is of a non-linear integro-differential type for which no closed solution is available. Furthermore, in the case of most chemically-active gases such as high-temperature air, there is no closed formulation for either k, κ or ρ since the latter, at constant pressure, is a function of the compressibility Z. Both k and Z are tabulated for high-temperature air in [12]. For these two reasons, the solution of equation (34) is only possible by iteration, like most other problems of radiative transfer.

This iteration is performed without difficulty when equation (34) is integrated and the two boundary conditions of given temperatures at the walls are used. The process will converge for physical reasons, if the temperature profile used as the first approximation is that due to conduction alone, provided the optical thickness τ_2 has a physical meaning.

It is possible to convert the "optical" solution $B(\tau)$ into a "physical" solution $T(y)$ by using the definition (33) of B and τ:

$$T = \left(\frac{B\pi}{\sigma}\right)^{1/4} \text{ and } y = \int_0^\tau \frac{dt}{\rho\kappa} \qquad (35)$$

Note that the correspondence $y - \tau$ is to be established only after the temperature profile $T(\tau)$ has been established, because of the dependence of κ on temperature. This point is of special importance since only a certain class of optical lengths τ have a physical correspondence y in this problem [13].

RESULTS AND CONCLUSIONS

Two numerical cases are presented on Figs. 4 and 5 for low and high values of T_0, respectively. On each diagram is shown the profile that would be obtained without radiation (which was used as the first try in the iteration) as well as the profiles obtained including radiation for lower wall emissivities $\varepsilon_0 = 0$, $\frac{1}{2}$ and 1. The following general remarks can be made:

1. *Whenever the wall effects are small* ($\varepsilon_0 = 0$, *or* $\varepsilon_0 \neq 0$ *but* $T_0 \ll T_2$)

To insure a constant total flux of energy across the gas layer, it is necessary for the convective flux variations to be compensated by opposite variations of the radiative flux.

Now, the radiation emitted by the gas layer near both limiting planes (*no wall effects considered*), is directed towards the outside, in the upper direction near the upper wall and in the lower direction near the lower wall. It is then expected that this radiative flux reversal across the layer will be compensated by an increased convective flux near the hot wall and by a decreased one near the cool wall. This is illustrated in Figs. 4 and 5 for the case $\epsilon_0 = 0$ (no wall effects): Larger temperature gradients are introduced by radiation near the hot wall (as compared to the purely conductive case) while smaller temperature gradients are found near the cool wall. *Convection to a cool wall is therefore reduced if the layer radiates.* As can be seen on Fig. 4, this conclusion also applies practically for all possible emissivities of the cool wall, because of its relatively low $B(0)$.

Although the additional radiative transfer makes for a higher total heat flux through the flow, it is therefore apparent that calculating this additional radiative flux to the cool wall by simply using the non-radiative temperature profile would lead in this case to an excessive

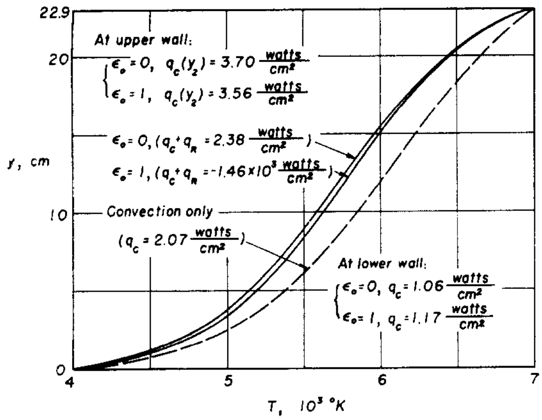

FIG. 4. Temperature profiles and fluxes of energy in air ($p = 0{\cdot}1$ atm). (Transparent upper wall, opaque lower wall.)

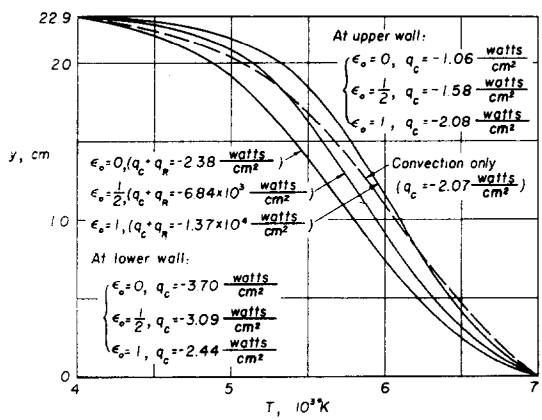

FIG. 5. Temperature profiles and fluxes of energy in air ($p = 0{\cdot}1$ atm). (Transparent upper wall, opaque lower wall.)

value for both convective and radiative fluxes. If, in addition, the lower cool wall is perfectly reflective, the total heat transfer to the wall (q_c alone, in this case) is effectively reduced by the presence of radiation.

2. *Whenever the wall effects are important* ($\epsilon_0 \neq 0$ *and* $T_0 \gg T_2$)

A fraction of the energy radiated by the hot lower wall is absorbed by each gas layer $2\pi\rho\kappa\epsilon_0 B(T_0)dy$; hence, a continuous decrease of the wall flux on its path upwards on Fig. 5. This contribution can be conveniently broken down into a constant flux $\epsilon_0 B(T_0)$ across the layer, minus a flux in the downward direction

$$2\pi \int_0^T \epsilon_0\, B(T_0)\, dt$$

which increases from zero at the lower wall to a maximum value at the upper wall. This arrangement is useful, because it is seen from equation (32) that only the variable part of the fluxes affects the temperature distribution.

This variable downwards flux is often larger than the variable flux due to the radiation of the gas layers themselves:

$$4\pi \int_0^T B(T)\, dt$$

This is especially true for the cooler layers where $\epsilon_0 B(T_0) > B(T)$ for most values of ϵ_0 and T_2; in this case, these two radiative effects, which determine alone the temperature profile in the problem, introduce a net radiative flux *downwards*. In opposition to the case where wall effects are negligible, the temperature gradients at the *cool wall* are therefore *larger* than they are without radiation (as can be seen in Fig. 5): this adjustment of the temperature profile compensates in part the net downward radiative flux at the upper cool wall by a larger upward convective flux. Another compensation comes from a reduction of the sum of these two variable fluxes as can be seen at the lower wall, where larger emissivities correspond to lower temperature gradients and convection rates than for $\epsilon_0 = 0$.

3. *In general*

An increased layer thickness will increase the role of radiation while decreasing the temperature gradient and thus the convection flux. Inversely, radiation effects will be relatively unimportant in very thin layers. The problems of the type considered here, correspond to the intermediate case where convection and radiation are of the same order of magnitude.

It is convenient to characterize these three classes of problems by writing an equation in non-dimensional form:

$$\frac{A(\tau_2)}{\tau_2^2}(B)\bar{A}\,\frac{d\bar{B}}{d\bar{\tau}} - $$

$$- 4\pi \int_0^{\bar{\tau}} [\bar{B}(t) - \frac{\epsilon_0}{2}\bar{B}(0)]\, dt = \text{const.} \qquad (36)$$

where

$$\bar{A}(B) = \frac{A\ (B)}{A\ (\tau_2)}\ ,\ \bar{B} = \frac{B}{B\ (\tau_2)} \text{ and } \bar{\tau} = \frac{\tau}{\tau_2}$$

In equation (36), the following ratio plays a special role:

$$N_{r-c} \equiv \frac{A\ (\tau_2)}{\tau_2^2} = \left| \frac{k}{\rho\kappa y^2\ 4\sigma\ T^3} \right|_{\tau=\tau_2} \qquad (37)$$

The magnitude of this non-dimensional parameter N_{r-c} (r–c for radiation–convection) determines the relative role of the convective term (the first term of equation (36)) vs. the radiative terms. For very large values of N_{r-c}, convection is the only appreciable transport process while radiation is the important process for low values of N_{r-c}.

A physical interpretation of N_{r-c} is also:

$$N_{r-c} = \frac{k\ (T_2/L)}{2\epsilon_2\sigma\ T_2^4} \qquad (38)$$

In equation (38), the numerator is a typical conductive heat flux from the hot wall across the gas layer and the denominator is the radiative flux from the gas layer assumed arbitrarily at the hot-wall temperature. This parameter is of the same family as the non-dimensional quantities discussed in [7] and [9].

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